

# Group actions, $k$ -derivations and finite morphisms

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## Abstract

Let  $G$  be an affine algebraic group over an algebraically closed field  $k$  of characteristic zero. In this paper, we consider finite  $G$ -equivariant morphisms  $F : X \rightarrow Y$  of irreducible affine varieties. First we determine under which conditions on  $Y$  the induced map  $F^G : X//G \rightarrow Y//G$  of quotient varieties is also finite. This result is reformulated in terms of kernels of derivations on  $k$ -algebras  $A \subset B$  such that  $B$  is integral over  $A$ . Second we construct explicitly two examples of finite  $G$ -equivariant maps  $F$ . In the first one,  $F^G$  is quasifinite but not finite. In the second one,  $F^G$  is not even quasifinite.

## 1 Introduction

We begin by recalling a few definitions from Geometric Invariant Theory; for more details, see [B-S]. Let  $G$  be a connected affine algebraic group over an algebraically closed field  $k$  of characteristic zero. An affine  $G$ -variety  $X$  is an affine variety together with an algebraic action of  $G$ , i.e. a regular map  $\varphi : G \times X \rightarrow X$ ,  $(g, x) \mapsto g.x$  that also defines an action of  $G$  on  $X$ . A  $G$ -equivariant morphism  $F : X \rightarrow Y$  of affine  $G$ -varieties is a regular map such that  $F(g.x) = g.F(x)$  for all  $(g, x) \in G \times X$ . For any affine  $G$ -variety  $X$ , denote by  $\mathcal{O}_X^G$  the ring of invariants of  $G$ . The algebraic quotient  $X//G$  is defined as the scheme:

$$X//G = \operatorname{Spec}(\mathcal{O}_X^G)$$

This notion is directly related to Hilbert's fourteenth Problem in Geometric Invariant Theory (see [Van]). It is well-known that  $X//G$  is affine if  $G$  is linearly reductive (see [B-S]) but that it neednot be affine in general (see for instance [Na] or [D-F]). Let  $F : X \rightarrow Y$  be a  $G$ -equivariant morphism of affine  $G$ -varieties. Since  $F^*(\mathcal{O}_Y^G) \subset \mathcal{O}_X^G$ , the map  $F$  induces a natural morphism  $F^G : X//G \rightarrow Y//G$ . In this paper, we are interested in the following question:

**Question 1.1** *If  $F$  is a finite morphism, under which conditions is  $F^G$  also finite?*

In [Van], p. 227, Van den Essen gave an example of a  $G_a$ -equivariant finite morphism  $F : k^3 \rightarrow C \times k^2$ , where  $C$  is a cuspidal curve. In this example, the quotient  $k^3//G_a$  is affine but  $C \times k^2//G_a$  is not, from which follows that  $F^{G_a}$  is not finite. Note that, in the case of a unipotent group  $G$ , the map  $F^G$  is always quasifinite (see lemma 3.1)

In any case, the Lie algebra  $\mathfrak{g}$  of  $G$  acts like a collection of  $k$ -derivations on  $\mathcal{O}_X$ . Since  $k$  has characteristic zero, the ring  $\mathcal{O}_X^G$  coincides with the kernel of this collection (see [Kr]). This invites us to reformulate question 1.1 in a more general setting. Consider two integral  $k$ -algebras  $A, B$  of finite type, where  $\text{char}(k) = 0$  and  $A \subset B$ . Let  $\mathcal{F}$  be any family of  $k$ -derivations on  $B$  which preserve  $A$ , i.e.  $d(A) \subset A$  for any  $d \in \mathcal{F}$ . Denote by:

$$A^{\mathcal{F}} = \bigcap_{d \in \mathcal{F}} \ker d|_A \quad \quad B^{\mathcal{F}} = \bigcap_{d \in \mathcal{F}} \ker d|_B$$

**Question 1.2** *If  $B$  is integral over  $A$ , under which conditions is  $B^{\mathcal{F}}$  integral over  $A^{\mathcal{F}}$ ?*

Note that, if  $A, B$  are fields instead of finitely generated  $k$ -algebras, then  $B^{\mathcal{F}}$  is obviously algebraic, hence integral over  $A^{\mathcal{F}}$ . This can be easily proved by using Differential Galois Theory (see [Ko]).

In this paper, we are going to give an answer to these questions, in terms of the properties of the scheme  $\text{Spec}(A)$ , and then produce some new counterexamples. We begin with some definitions. Let  $A$  be a finitely generated  $k$ -algebra. A closed point  $x$  of  $\text{Spec}(A)$  (considered as a maximal ideal of  $A$ ) is a *non normal singular point* if the local ring  $A_x$  is not integrally closed in its fraction field. In this case,  $\text{Spec}(A)$  is not smooth at  $x$ . However, any singularity does not have to be non normal. For instance, the surface  $S$  in  $\mathbb{C}^3$  given by the equation  $xz - y^2 = 0$  has a unique normal singularity at the origin.

**Definition 1.3** *A finitely generated  $k$ -algebra  $A$  has isolated non normal singular points if the set of non normal singular points of  $\text{Spec}(A)$  is finite.*

**Theorem 1.4** *Let  $A \subset B$  be two finitely generated integral  $k$ -algebras, where  $\text{char}(k) = 0$ . Let  $\mathcal{F}$  be a family of  $k$ -derivations on  $B$  such that  $d(A) \subset A$  for any  $d \in \mathcal{F}$ . If  $B$  is integral over  $A$  and if  $A$  has isolated non normal singular points, then  $B^{\mathcal{F}}$  is integral over  $A^{\mathcal{F}}$ .*

This result is the best one can expect, regarding the singularities of  $\text{Spec}(A)$ . More precisely, there exist examples of actions on varieties with as few singularities as possible, and for which the conclusion of theorem 1.4 fails. We construct such examples as follows. For any positive integers  $n, m$ , set  $B_{n,m} = k[x_1, \dots, x_n, y_1, \dots, y_m, z]$ , and define  $A_{n,m}$  as the  $k$ -subalgebra of  $B_{n,m}$  generated by:

- the monomials  $y_1, \dots, y_m, z$ ,
- the polynomials  $x_i^2 + x_i z$  and  $x_i^3 + x_i^2 z$ , for any  $i = 1, \dots, n$ ,
- the monomials  $x_1^{i_1} \dots x_n^{i_n} y_j$ , where  $j = 1, \dots, m$  and  $i_k \leq 1$  for any  $k = 1, \dots, n$ .

Note that  $B_{n,m}$  is integral over  $A_{n,m}$ . Indeed, the monomials  $y_1, \dots, y_m, z$  belong to  $A_{n,m}$ . If  $t_i = x_i^2 + x_i z$ , then  $t_i$  belongs to  $A_{n,m}$  and each  $x_i$  satisfies the relation  $x_i^2 + z x_i - t_i = 0$ . Since  $B_{n,m}$  is normal and has the same fraction field as  $A_{n,m}$ ,  $B_{n,m}$  is the integral closure of  $A_{n,m}$ . We set  $X_{n,m} = k^{n+m+1}$  and  $Y_{n,m} = \text{Spec}(A_{n,m})$ . The inclusion  $A_{n,m} \subset B_{n,m}$  induces the so-called normalization morphism:

$$F : X_{n,m} \longrightarrow Y_{n,m}$$

which is finite. Let  $G_1$  be the additive group  $G_a(k)$  and  $G_2$  the group  $\text{Aut}(k)$  of automorphisms of the line, i.e. the set of morphisms of the form  $z \mapsto az + b$ , where  $a \neq 0$ . We endow  $X_{n,m}$  with two algebraic actions  $\varphi, \psi$  of  $G_1, G_2$  respectively, defined by the formulas:

$$\begin{aligned} \varphi_t(x_1, \dots, x_n, y_1, \dots, y_m, z) &= (x_1, \dots, x_n, y_1, \dots, y_m, z + t y_1) \\ \psi_{(a,b)}(x_1, \dots, x_n, y_1, \dots, y_m, z) &= (x_1, \dots, x_n, a y_1, \dots, a y_m, z + b y_1) \end{aligned}$$

Since the morphisms  $\varphi_t^*$  and  $\psi_{(a,b)}^*$  preserve the ring  $A_{n,m}$ ,  $\varphi$  and  $\psi$  induce two actions of  $G_1$  and  $G_2$  on the variety  $Y_{n,m}$ . Moreover, the map  $F$  is equivariant for both  $G_1$  and  $G_2$ . It is then easy to check that:

$$B_{n,m}^{G_1} = k[x_1, \dots, x_n, y_1, \dots, y_m] \quad \text{and} \quad B_{n,m}^{G_2} = k[x_1, \dots, x_n]$$

**Theorem 1.5** *Let  $X_{n,m}, Y_{n,m}, \varphi$  and  $\psi$  be the varieties and actions defined above. Then the singular set of  $Y_{n,m}$  has dimension  $\leq n$ . Moreover the actions  $\varphi$  and  $\psi$  enjoy the following properties:*

- $A_{n,m}^{G_1} = k[x_1^{i_1} \dots x_n^{i_n} y_1^{j_1} \dots y_m^{j_m}, j_1 + \dots + j_m > 0]$ . In particular,  $A_{n,m}^{G_1}$  is not finitely generated,  $B_{n,m}^{G_1}$  is algebraic but not integral over  $A_{n,m}^{G_1}$  and  $F^{G_1}$  is quasifinite but not finite.
- $A_{n,m}^{G_2} = k$ . In particular,  $B_{n,m}^{G_2}$  is not algebraic over  $A_{n,m}^{G_2}$ , the transcendence degree of  $B_{n,m}^{G_2}$  over  $A_{n,m}^{G_2}$  is equal to  $n$  and  $F^{G_2}$  is not even quasifinite.

First note that the variety  $Y_{1,m}$  has dimension  $m+2$  and its singular set  $\text{Sing}(Y_{1,m})$  is a line. In particular, this line consists solely of non normal singular points, and this shows that theorem 1.4 is optimal in terms of singularities. In Van den Essen's example, the second variety has a singular set of dimension 2. Second note that the difference of dimension between the quotient varieties for the action of  $G_2$  is equal to  $n$ , hence it can be chosen arbitrarily large.

## 2 Proof of theorem 1.4

In this section, we are going to give a proof of theorem 1.4, first in the case when  $A$  is a normal ring, and then in the general case. We begin by recalling an elementary result from Differential Galois Theory, which can be found for instance in [Ko].

**Lemma 2.1** *Let  $k \subset K \subset L$  be fields of characteristic zero, such that the extension  $L/K$  is finite. If  $d$  is a  $k$ -derivation on  $K$ , then  $d$  extends uniquely to a  $k$ -derivation  $D$  of  $L$ .*

**Lemma 2.2** *Let  $A \subset B$  be two finitely generated integral  $k$ -algebras, where  $\text{char}(k) = 0$ , such that  $B$  is integral over  $A$ . Let  $\mathcal{F}$  be a family of  $k$ -derivations on  $B$  such that  $d(A) \subset A$  for any  $d$  in  $\mathcal{F}$ . If  $A$  is normal, then  $B^{\mathcal{F}}$  is integral over  $A^{\mathcal{F}}$ .*

*Proof:* Denote by  $K$  the fraction field of  $A$ , and by  $K'$  that of  $B$ . Since  $B$  is integral over  $A$  and that  $A, B$  are finitely generated  $k$ -algebras,  $B$  is a finite  $A$ -module. In particular,  $K'/K$  is a finite extension and  $k$  is contained in both  $K$  and  $K'$ . Denote by  $L/K$  a finite Galois extension containing  $K'/K$ , with Galois group  $G$ . Such an extension exists since  $K$  has characteristic zero. For any element  $x$  of  $B^{\mathcal{F}}$ , consider the polynomial:

$$P(t) = \prod_{g \in G} (t - g(x))$$

It is clear that the coefficients of  $P$  are invariant with respect to  $G$ , hence they belong to  $K$ . Since  $x$  is integral over  $A$ ,  $g(x)$  is integral over  $A$  for all  $g \in G$ , and the coefficients of  $P$  are integral over  $A$ . Since they all belong to  $K$ , and that  $A$  is normal, they all lie in  $A$ . There remains to show that the coefficients of  $P$  are annihilated by every element of  $\mathcal{F}$ .

Let  $d$  be any derivation belonging to  $\mathcal{F}$ . Then  $d$  defines a  $k$ -derivation on  $K$ . Denote by  $D$  its unique extension to  $L$  (see lemma 2.1). For any  $g \in G$ , consider the map:

$$D_g = g^{-1} \circ D \circ g$$

Since  $k \subset K$  and  $K$  is  $G$ -invariant,  $D_g$  is  $k$ -linear and  $D_g$  coincides with  $d$  on  $K$ . Moreover  $D_g$  is a  $k$ -derivation on  $L$ . Indeed, for any  $x, y \in L$ , we have:

$$\begin{aligned} D_g(xy) &= g^{-1} \circ D(g(x)g(y)) \\ &= g^{-1}(g(x)D(g(y)) + g(y)D(g(x))) \\ &= xg^{-1}D(g(y)) + yg^{-1}D(g(x)) \\ &= xD_g(y) + yD_g(x) \end{aligned}$$

By uniqueness of the extension,  $D = D_g$  on  $L$ . In particular,  $D \circ g = g \circ D$  for any  $g \in G$ . Since  $D(x) = d(x) = 0$ , we find  $D(g(x)) = 0$  for all  $g \in G$ . So the coefficients of  $P$  all lie in the kernel of  $d$ . Since this holds for any derivation in  $\mathcal{F}$ , these coefficients all belong to  $A^{\mathcal{F}}$  and the result follows. ■

**Lemma 2.3** *Let  $A$  be a finitely generated integral  $k$ -algebra. Let  $A'$  be its integral closure. If  $A$  has isolated non normal singular points, then  $A'/A$  is a finite dimensional  $k$ -vector space.*

*Proof:* Let  $\{x_1, \dots, x_n\}$  be the collection of non normal singular points of  $\text{Spec}(A)$ , viewed as maximal ideals of  $A$ , and set  $I = x_1 \cap \dots \cap x_n$ . First we claim that  $A/I$  is finite dimensional

over  $k$ . Indeed since the  $x_i$  are maximal ideals, we obtain by the Chinese Remainder Theorem:

$$\frac{A}{I} \simeq \frac{A}{x_1} \times \dots \times \frac{A}{x_n}$$

Each  $k$ -algebra  $A/x_i$  is finitely generated and is a field, hence it is a finite extension of  $k$  (see [Hu]). In particular, every quotient  $A/x_i$  has finite dimension, and  $\dim_k A/I < +\infty$ .

Second we show that  $A/I^m$  is finite dimensional over  $k$  for any  $m > 0$ . The  $A$ -module  $A/I^m$  is filtered by the submodules  $M_i = I^i/I^m$  for  $i = 0, \dots, m$ , and  $M_i/M_{i+1} = I^i/I^{i+1}$ . Since  $A$  is noetherian and  $I$  is an ideal of  $A$ ,  $M_i/M_{i+1}$  is a finite  $A/I$ -module, hence finite dimensional over  $k$ . Therefore we have  $\dim_k A/I^m < +\infty$ .

Eventually we prove that  $A'/A$  is finite dimensional over  $k$ . Let  $f_1, \dots, f_r$  be a set of nonzero generators of  $I$ . For any  $f_i$ , the localization  $A_{(1/f_i)}$  has no non normal singular points, hence it is a normal ring and  $A \subset A' \subset A_{(1/f_i)}$ . Since  $A'$  is a finite  $A$ -module, there exists an integer  $n_i$  such that  $f_i^{n_i} A' \subset A$ . If  $m = r \max\{n_i\}$ , then  $I^m A' \subset A$  and  $I^m(A'/A) = 0$ . So  $A'/A$  is a finite  $A/I^m$ -module. Since  $\dim_k A/I^m < +\infty$ , the result follows. ■

*Proof of theorem 1.4:* Let  $A \subset B$  be two finitely generated integral  $k$ -algebras, where  $\text{char}(k) = 0$ , such that  $B$  is integral over  $A$ . Let  $\mathcal{F}$  be a family of  $k$ -derivations on  $B$  such that  $d(A) \subset A$  for any  $d$  in  $\mathcal{F}$ . Assume that  $A$  has isolated non normal singular points. Let us prove that  $B^\mathcal{F}$  is integral over  $A^\mathcal{F}$ .

Let  $K$  be the fraction field of  $B$ , and consider the  $k$ -subalgebra  $B'$  of  $K$  generated by  $B$  and the integral closure  $A'$  of  $A$ . Since  $A, B$  are finitely generated,  $A'$  and  $B'$  are also finitely generated. By construction,  $B'$  is integral over  $A'$ . Moreover, each  $k$ -derivation  $d \in \mathcal{F}$  extends to a unique  $k$ -derivation on  $A'$  by Seidenberg's Theorem (see [Sei]). Every derivation  $d$  is also well-defined on  $K$ . Since  $d(A') \subset A'$  and  $d(B) \subset B$ , we have  $d(B') \subset B'$  for any  $d \in \mathcal{F}$ . Since  $A'$  is normal,  $(B')^\mathcal{F}$  is integral over  $(A')^\mathcal{F}$  by lemma 2.2. Since  $B^\mathcal{F} \subset (B')^\mathcal{F}$ , there only remains to show that  $(A')^\mathcal{F}$  is integral over  $A^\mathcal{F}$ . Let  $x$  be any element of  $(A')^\mathcal{F}$ . Since  $A$  has isolated non normal singular points,  $A'/A$  is finite dimensional by lemma 2.3. In particular, there exist some elements  $a_0, \dots, a_{n-1} \in k$  such that:

$$P(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0 \in A$$

By construction,  $P(x)$  is annihilated by all elements of  $\mathcal{F}$ . So  $P(x)$  belongs to  $A^\mathcal{F}$  and  $x$  satisfies an integral relation with coefficients in  $A^\mathcal{F}$ . Since this holds for any  $x$  in  $(A')^\mathcal{F}$ ,  $(A')^\mathcal{F}$  is integral over  $A^\mathcal{F}$  and the result follows. ■

### 3 A lemma on unipotent group actions

**Lemma 3.1** *Let  $G$  be a unipotent algebraic group over an algebraically closed field  $k$ . Let  $F : X \rightarrow Y$  be a finite morphism of affine irreducible  $G$ -varieties. Then  $F^G$  is quasifinite.*

*Proof:* Set  $B = \mathcal{O}_X$  and  $A = \mathcal{O}_Y$ . The morphism  $F$  induces an inclusion  $A \subset B$  such that  $B$  is integral over  $A$ . Choose an element  $x$  of  $B$  which is  $G$ -invariant, and let  $P(x) = a_n t^n + \dots + a_0$  be a polynomial in  $A[t]$ , of minimal degree  $n$  such that  $P(x) = 0$ . Consider the subset  $M$  of  $A^{n+1}$  consisting of the  $(n+1)$ -tuples  $(a_0, \dots, a_n)$  such that  $a_n x^n + \dots + a_0 = 0$ . By construction,  $M$  is an  $A$ -submodule of  $A^{n+1}$ , and  $A^{n+1}$  is endowed with the action of  $G$  defined by:

$$g \cdot (a_0, \dots, a_n) = (g \cdot a_0, \dots, g \cdot a_n)$$

Since  $x$  is  $G$ -invariant,  $M$  is a rational  $G$ -submodule of  $A^{n+1}$ . Now since  $G$  is unipotent, there exists a nonzero element  $b = (b_0, \dots, b_n)$  of  $M$  which is  $G$ -invariant (see [B-S]). In particular, all the  $b_i$  are  $G$ -invariant and we have:

$$b_n x^n + \dots + b_0 = 0$$

Note that  $b_n$  cannot be equal to zero. Otherwise, all the  $b_i$  would be zero by minimality of  $n$ , a contradiction. So  $x$  is algebraic over  $A^G$ . Since this holds for any  $x$  in  $B^G$ ,  $B^G$  is algebraic over  $A^G$ . In particular, the morphism  $F^G$  is quasi-finite. ■

## 4 Properties of the varieties $Y_{n,m}$

In this section, we are going to establish theorem 1.5. We will begin with a few lemmas concerning the ring  $A_{n,m}$  defined in the introduction.

### 4.1 A few preliminary lemmas

**Lemma 4.1**  $k[x_1^{i_1} \dots x_n^{i_n} y_1^{j_1} \dots y_m^{j_m}, j_1 + \dots + j_m > 0] \subset A_{n,m}$ .

*Proof:* We are going to prove by induction on  $p = i_1 + \dots + i_n$  that every monomial of the form  $x_1^{i_1} \dots x_n^{i_n} y_1^{j_1} \dots y_m^{j_m}$ , where  $j_1 + \dots + j_m > 0$ , belongs to  $A_{n,m}$ . Since  $y_1, \dots, y_m$  belong to  $A_{n,m}$ , we may restrict ourselves to the monomials of the form  $x_1^{i_1} \dots x_n^{i_n} y_j$ , where  $j = 1, \dots, m$ . For  $p = 0$ , this is clear because  $A_{n,m}$  contains  $y_1, \dots, y_m$ . Assume the property holds to the order  $p$ . For convenience, we set  $t_i = x_i^2 + x_i z$  and note that every  $t_i$  belongs to  $A_{n,m}$ . Consider any monomial  $a$  of the form  $x_1^{i_1} \dots x_n^{i_n} y_j$ , where  $i_1 + \dots + i_n = p + 1$ . If all the  $i_k$  are  $\leq 1$ , then  $a$  belongs to  $A_{n,m}$  by construction. If one of the  $i_k$  is  $\geq 2$ , for instance  $i_1 \geq 2$ , then write  $a = x_1^2 b$ , where  $b = x_1^{i_1-2} \dots x_n^{i_n} y_j$  and set  $c = x_1^{i_1-1} \dots x_n^{i_n} y_j$ . Since  $x_1^2 + x_1 z - t_1 = 0$ , we obtain by multiplication by  $b$ :

$$a + zc - t_1 b = 0$$

By our induction's hypothesis, the monomials  $b$  and  $c$  belong to  $A_{n,m}$ . So  $a$  belongs to  $A_{n,m}$  and the result follows. ■

**Lemma 4.2**  $k[x_1, \dots, x_n, y_1, \dots, y_m] \cap A_{n,m} = k[x_1^{i_1} \dots x_n^{i_n} y_1^{j_1} \dots y_m^{j_m}, j_1 + \dots + j_m > 0]$ .

*Proof:* Let  $f$  be an element of  $k[x_1, \dots, x_n, y_1, \dots, y_m] \cap A_{n,m}$ . By the previous lemma, we know that every monomial of  $k[x_1, \dots, x_n, y_1, \dots, y_m]$  containing at least one of the  $y_j$  belongs to  $A_{n,m}$ . Up to subtracting a linear combination of such monomials to  $f$ , we may assume that  $f$  only depends on  $x_1, \dots, x_n$ . If we show that such an  $f$  is always constant, then the lemma will follow. So we are going to prove by induction on  $n \geq 1$  that any element  $f$  of  $A_{n,m}$  that only depends on  $x_1, \dots, x_n$  is a constant. For  $n = 1$ , consider such an element  $f = f(x_1)$  of  $A_{1,m}$ . Then there exists a polynomial  $P$  such that:

$$f(x_1) = P(y_1, \dots, y_m, z, x_1^2 + zx_1, x_1^3 + zx_1^2, x_1y_1, \dots, x_1y_m)$$

By setting  $y_1 = \dots = y_m = 0$ , we can see there exists a polynomial  $Q$  such that:

$$f(x_1) = Q(z, x_1^2 + zx_1, x_1^3 + zx_1^2)$$

If  $x_1 = 0$ , then  $Q(z, 0, 0) = f(0)$  is a constant. So the polynomial  $Q(a, b, c) - f(0)$  has no pure terms in  $a$ , and it can be expanded as:

$$Q(a, b, c) - f(0) = \sum_{k+l>0} Q_{k,l}(a) b^k c^l$$

In particular, this yields for  $f(x_1) - f(0)$ :

$$f(x_1) - f(0) = \sum_{k+l>0} Q_{k,l}(z) (x_1^2 + zx_1)^k (x_1^3 + zx_1^2)^l$$

Since  $x_1 + z$  divides both  $x_1^2 + zx_1$  and  $x_1^3 + zx_1^2$ ,  $x_1 + z$  must divide  $f(x_1) - f(0)$ , which is impossible unless  $f$  is constant. Now assume the property holds to the order  $(n - 1)$ , and let  $f = f(x_1, \dots, x_n)$  be an element of  $A_{n,m}$ . Then there exists a polynomial  $P$  such that:

$$f(x_1, \dots, x_n) = P(y_1, \dots, y_m, z, x_1^2 + zx_1, x_1^3 + zx_1^2, \dots, x_n^2 + zx_n, x_n^3 + zx_n^2, \dots, x_1^{i_1} \dots x_n^{i_n} y_j, \dots)$$

By setting  $y_1 = \dots = y_m = 0$ , we can see there exists a polynomial  $Q$  such that:

$$f(x_1, \dots, x_n) = Q(z, x_1^2 + zx_1, x_1^3 + zx_1^2, \dots, x_n^2 + zx_n, x_n^3 + zx_n^2)$$

By setting  $x_n = 0$ , we find:

$$f(x_1, \dots, x_{n-1}, 0) = Q(z, x_1^2 + zx_1, x_1^3 + zx_1^2, \dots, x_{n-1}^2 + zx_{n-1}, x_{n-1}^3 + zx_{n-1}^2, 0, 0)$$

So the polynomial  $f(x_1, \dots, x_{n-1}, 0)$  belongs to  $A_{n-1,m}$ . By our induction's hypothesis,  $f(x_1, \dots, x_{n-1}, 0)$  is constant, and we may assume that:

$$Q(a, b_1, c_1, \dots, b_{n-1}, c_{n-1}, 0, 0) - f(0, \dots, 0) = 0$$

In particular,  $Q(a, b_1, c_1, \dots, b_n, c_n) - f(0, \dots, 0)$  has no pure terms in  $a, b_1, c_1, \dots, b_{n-1}, c_{n-1}$ , and it can be expanded as:

$$Q(a, b_1, c_1, \dots, b_n, c_n) - f(0, \dots, 0) = \sum_{k+l>0} Q_{k,l}(a, b_1, c_1, \dots, b_{n-1}, c_{n-1}) b_n^k c_n^l$$

If  $q_{k,l} = Q_{k,l}(z, x_1^2 + zx_1, x_1^3 + zx_1^2, \dots, x_{n-1}^2 + zx_{n-1}, x_{n-1}^3 + zx_{n-1}^2)$ , then this yields:

$$f(x_1, \dots, x_n) - f(0, \dots, 0) = \sum_{k+l>0} q_{k,l}(x_n^2 + zx_n)^k (x_n^3 + zx_n^2)^l$$

Since  $x_n + z$  divides both  $x_n^2 + zx_n$  and  $x_n^3 + zx_n^2$ ,  $x_n + z$  must divide  $f(x_1, \dots, x_n) - f(0, \dots, 0)$ , which is impossible unless  $f$  is constant, and the result follows. ■

**Lemma 4.3** *The  $k$ -algebra  $\mathcal{A}_{n,m} = k[x_1^{i_1} \dots x_n^{i_n} y_1^{j_1} \dots y_m^{j_m}, j_1 + \dots + j_m > 0]$  is not finitely generated.*

*Proof:* Suppose on the contrary that  $\mathcal{A}_{n,m}$  is finitely generated, and let  $f_1, \dots, f_r$  be a system of generators. For convenience, we may assume that  $f_i(0, \dots, 0) = 0$  for any  $i$ . Since every  $f_i$  is a linear combination of monomials of the form  $x_1^{i_1} \dots x_n^{i_n} y_1^{j_1} \dots y_m^{j_m}$ , where  $j_1 + \dots + j_m > 0$ , we may even assume that  $\{f_1, \dots, f_r\}$  consists solely of such monomials. Then for any couple of integers  $i_1, j_1$ , where  $j_1 > 0$ , there exists a polynomial  $P$  such that:

$$x_1^{i_1} y_1^{j_1} = P(f_1, \dots, f_r)$$

Let  $g_i$  be the monomial  $f_i(x_1, y_1, 0, \dots, 0)$  for any  $i$ . Then, it is easy to check that  $g_1, \dots, g_r$  belong to  $\mathcal{A}_{1,1}$ . By setting  $x_2 = \dots = x_n = y_2 = \dots = y_m = 0$ , we find:

$$x_1^{i_1} y_1^{j_1} = P(g_1, \dots, g_r)$$

In particular,  $g_1, \dots, g_r$  span the  $k$ -algebra  $\mathcal{A}_{1,1}$ . Set  $g_i = x_1^{n_i} y_1^{m_i}$  for any  $i$ , and consider the monomial  $x_1^s y_1$ , where  $s > \max\{n_i\}$ . Since  $x_1^s y_1$  belongs to  $\mathcal{A}_{1,1}$ , there exists a polynomial  $P$  such that:

$$x_1^s y_1 = P(g_1, \dots, g_r)$$

Since the  $g_i$  are monomials,  $P$  must contain a monomial of the form  $u_1^{a_1} \dots u_r^{a_r}$  such that:

$$\begin{aligned} s &= a_1 n_1 + \dots + a_r n_r \\ 1 &= a_1 m_1 + \dots + a_r m_r \end{aligned}$$

Since the  $a_i$  are nonnegative integers and that  $m_i > 0$  for any  $i$ , this implies that  $a_i$  is zero for every index  $i$  except one, say  $i_0$ , and that  $a_{i_0} = 1$ . But then  $s = n_{i_0}$ , which is impossible since  $s > n_i$  for any  $i$ . ■



## 4.2 Proof of theorem 1.5

Let  $Y_{n,m}$  be the variety defined in the introduction. We first show that  $Sing(Y_{n,m})$  has dimension  $\leq n$ . If we localize  $A_{n,m}$  with respect to any  $y_i$ , then we get the ring:

$$(A_{n,m})_{\frac{1}{y_i}} = k[x_1, \dots, x_n, y_1, \dots, y_i, \frac{1}{y_i}, \dots, y_m, z]$$

which is obviously smooth. Again by localizing  $A_{n,m}$  with respect to either  $x_1^{i_1} \dots x_n^{i_n} y_j$ , or  $(x_1^2 + x_1 z) \dots (x_n^2 + x_n z)$ , we get a regular ring. This implies that  $Sing(Y_{n,m})$  is contained in the zero set of the ideal  $I$  generated by all these polynomials. In particular,  $Sing(Y_{n,m})$  is contained in the image by  $F$  of the zero set  $E$  of all these polynomials in  $k^{n+m+1}$ . This set  $E$  consists of a finite union of sets of the form  $V(y_1, \dots, y_n, x_i)$  and  $V(y_1, \dots, y_n, x_i + z)$ . So  $E$  has dimension  $n$  and we get:

$$\dim Sing(X_{n,m}) \leq n$$

Second we compute the ring of invariants  $A_{n,m}^{G_1}$ . By construction, we clearly have  $B_{n,m}^{G_1} = k[x_1, \dots, x_n, y_1, \dots, y_m]$ . Since  $A_{n,m}$  is a subalgebra of  $B_{n,m}$ , we obtain by lemma 4.2:

$$A_{n,m}^{G_1} = A_{n,m} \cap k[x_1, \dots, x_n, y_1, \dots, y_m] = k[x_1^{i_1} \dots x_n^{i_n} y_1^{j_1} \dots y_m^{j_m}, j_1 + \dots + j_m > 0]$$

By lemma 4.3, this algebra is not finitely generated. We claim that  $B_{n,m}^{G_1}$  cannot be integral over  $A_{n,m}^{G_1}$ . Indeed, assume that  $B_{n,m}^{G_1}$  is integral over  $A_{n,m}^{G_1}$ . Then the  $x_i, y_j$  satisfy some integral relations over  $A_{n,m}^{G_1}$ . Let  $A_0$  be the  $k$ -algebra generated by all the coefficients of these relations. By construction,  $A_0$  is a finitely generated subalgebra of  $A_{n,m}^{G_1}$ , and  $B_{n,m}^{G_1}$  is integral over  $A_0$ . Since  $B_{n,m}^{G_1}$  is finitely generated,  $B_{n,m}^{G_1}$  is a finite  $A_0$ -module. But  $A_0$  is noetherian, and  $A_{n,m}^{G_1}$  is an  $A_0$ -submodule of  $B_{n,m}^{G_1}$ , so  $A_{n,m}^{G_1}$  is a finite  $A_0$ -module. If  $\{e_1, \dots, e_r\}$  denotes a basis of  $A_{n,m}^{G_1}$  over  $A_0$ , then:

$$A_{n,m}^{G_1} = A_0[e_1, \dots, e_r]$$

In particular,  $A_{n,m}^{G_1}$  is finitely generated, hence a contradiction. However,  $B_{n,m}^{G_1}$  is algebraic over  $A_{n,m}^{G_1}$ . Indeed, they have the same fraction field  $k(x_1, \dots, x_n, y_1, \dots, y_m)$ .

Consider now the ring  $B_{n,m}^{G_2}$ . The group  $G_2$  is the semi-direct product of  $G_a(k)$  and  $G_m(k)$ . The action of  $G_a(k)$  corresponds to the previous action  $\varphi$  of  $G_1$ . The action of  $G_m(k)$  is related to the weighted homogeneous degree  $deg$  on  $k[x_1, \dots, x_n, y_1, \dots, y_m, z]$ , which assigns the weight 0 to each  $x_i$  and  $z$ , and 1 to each  $y_i$ . In particular, the invariants of  $G_2$  on  $k[x_1, \dots, x_n, y_1, \dots, y_m, z]$  are the  $G_1$ -invariant polynomials of degree zero with respect to the  $y_i$ . More precisely:

$$B_{n,m}^{G_2} = k[x_1, \dots, x_n]$$

By lemma 4.2, the  $G_1$ -invariants of  $A_{n,m}$  of degree zero with respect to the  $y_i$  are the constants, i.e.  $A_{n,m}^{G_2} = k$ . In particular,  $B_{n,m}^{G_2}$  is not even algebraic over  $A_{n,m}^{G_2}$ , and the transcendence degree of  $B_{n,m}^{G_2}$  over  $A_{n,m}^{G_2}$  is equal to  $n$ . This ends the proof of theorem 1.5.

## References

- [B-S] M.Brion, G.Schwarz *Théorie des invariants et Géométrie des variétés quotients*, Collection Travaux en Cours 61, Hermann, Paris, 2000.
- [D-F] D.Daigle, G.Freudentburg *A counterexample to Hilbert's fourteenth problem in dimension 5*, J. Algebra 221 (1999), n°2, 528-535.
- [Ei] D.Eisenbud *Commutative Algebra with a view toward Algebraic Geometry*, Graduate Texts in Mathematics, 150, Springer Verlag New York, 1995.
- [Hu] J.Humphreys *Linear algebraic groups*, Graduate Texts in Mathematics n° 21, Springer Verlag New-York Heidelberg, 1975.
- [Ko] E.R.Kolchin *Differential Algebra and Algebraic Groups*, Pure and Applied Mathematics 54, Academic Press, New York London, 1973.
- [Kr] H.Kraft *Geometrische Methoden in der Invariantentheorie*, Aspects of Mathematics, D1 Friedr. Vieweg & Sohn, Braunschweig, 1984.
- [Na] M.Nagata *On the fourteenth problem of Hilbert*, 1960 Proc. Internat. Congress Math. 1958, pp. 459-462, Cambridge Univ.Press, New York.
- [Sei] A.Seidenberg *Derivations and integral closure*, Pacific J. Maths 16, 1966, 167-173.
- [Sh] I.Shafarevich *Basic Algebraic Geometry 1: Varieties in projective spaces*, second edition, Springer Verlag, Berlin, 1994.
- [Van] A.Van den Essen *Polynomial Automorphisms and the Jacobian Conjecture*, Progress in Maths 190, Birkhäuser Verlag, Basel 2000.